

# A SHORT INTRODUCTION TO SHADOWS OF 4-MANIFOLDS

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ABSTRACT. We give a self-contained introduction to the theory of shadows as a tool to study smooth 3 and 4-manifolds. The goal of the present paper twofold: on one side it is intended to be a shortcut to a basic use of the theory of shadows, on the other side it gives a sketchy overview of some of the most recent results on shadows. No original result is proved here and most of the details of the proofs are not included.

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## 1. INTRODUCTION

Shadows were defined by V. Turaev for the first time at the beginning of the nineties in [19] as a method for representing knots alternative to the standard one based on knot diagrams and Reidemeister moves. The theory was then developed in the preprint “Topology of shadows” which was later included in a revised form in [17]; moreover, a short account of the theory was published by Turaev in [18]. Since then, probably due to the slightly higher degree of complication of this theory with respect to Kirby calculus, only few applications of shadows were studied. Among these applications we recall the use of shadows to study Jones-Vassiliev invariants of knots made by U. Burri in [3] and A. Shumakovitch in [15] and the study of “Interdependent modifications of links and invariants of finite degree” developed by M.N. Goussarov in [10].

In the last two years, there has been a new surge of interest in shadows testified by the appearance of the preprint [16] of D. Thurston on knotted trivalent graphs and shadows and by the study of a new notion of complexity of 3 manifolds based on shadows developed

by the author and D. Thurston in [9]. This notion of “shadow complexity” turns out to be intimately connected with the geometric properties of 3-manifolds and in particular with their hyperbolic structures; this fact was also used in [4] to exhibit and study a particular class of 3-manifolds having strong geometric properties. Moreover a refinement of the notion of shadow of 4-manifold, called “branched shadow” has been studied by the author in [7], to attack a combinatorial study of gauge invariants and complex convexity problems. Branched shadows allow one to encode homotopy classes of almost complex structures on 4-manifolds, which, on manifolds admitting a shadow, are in a natural bijection with the set of  $Spin^c$ -structures. Moreover, it has been proved by the author that branched shadows embedded in complex manifolds behave like real surfaces do; in particular a classification of the points of a branched shadow into totally real, complex elliptic and complex hyperbolic ones is possible. These arguments were then used to study geometric problems as the existence of integrable representatives of homotopic classes of almost complex structures or the existence of Stein domain structures on the neighborhood of a shadow.

Roughly speaking, shadows of 4-manifolds are 2-dimensional polyhedra embedded in 4-manifolds equipped with extra-structures suitable to encode the topology of their regular neighborhood; they represent the analogous of spines of 3-manifolds in the 4-dimensional case. Shadows allow a completely combinatorial approach similar to the one used in the 3-dimensional case by means of spines.

Since the first obstacle to the comprehension of shadows is the familiarity with simple polyhedra, we inserted a first section in which we clarify what a simple polyhedron is and how it can be combinatorially constructed by means of a finite set of basic building blocks. Sections 3,4 and 5 are devoted to expose the basic facts of the theory of shadows proved by Turaev in [17]. In Section 3, we study how an embedding of a polyhedron into a 4-manifold can be used to equip the polyhedron with an additional structure, called “gleam” by Turaev, which is fundamental in the theory. In Section 4, we recall Turaev’s method to thicken a polyhedron equipped with gleams and to construct 4-manifolds. In Section 5, we introduce a set of basic moves on shadows and study their effect on the corresponding 4-manifolds.

The last three sections are devoted to an overview of some of the open problems and questions. In Section 6, we outline connections between shadows of 4-manifolds and the Andrews-Curtis conjecture. Section 7 is devoted to the results proven recently by the author and D. Thurston on shadow complexity and hyperbolic geometry of 3-manifolds. At last, in Section 8 we outline some of the recent developments of the theory towards a combinatorial method for studying 4-manifolds equipped with additional structures such as  $Spin^c$ , almost-complex, complex and Stein structures.

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## 2. SIMPLE POLYHEDRA

The basic objects underlying shadows of 4-manifolds are simple 2-dimensional polyhedra, that is 2-dimensional CW-complexes whose points have regular neighborhoods of “simple”

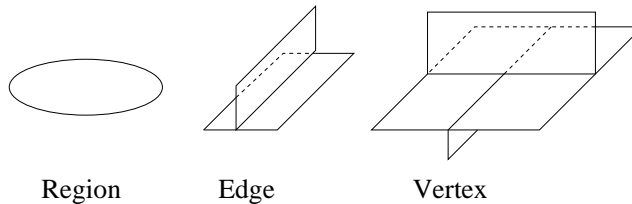


FIGURE 1. The local models of a simple polyhedron.

type as those shown in Figure 1. From the figure, we observe that the set of points whose regular neighborhood in a simple polyhedron  $P$  is shaped on the last two models, and which is called the *Singular set* of the polyhedron, is a 4-valent graph denoted by  $Sing(P)$  whose vertices are exactly the points whose regular neighborhoods are shaped on the third model and are called *vertices* of the polyhedron. The complement of the singular set of the polyhedron is the union of a set of open surfaces embedded in the polyhedron and called *regions*.

One check that the local pattern around a vertex is homeomorphic to the regular neighborhood of the vertex of the cone over the edges of a tetrahedron; this shows that the full symmetry group of the regular neighborhood of a vertex is the group of permutations of 4 elements. Another way of visualizing a vertex is to consider the result of attaching the polyhedron  $Y \times [0, 1]$  (where  $Y$  is a  $Y$ -shaped graph) along  $Y \times 0$  on a  $Y$ -shaped graph embedded in a disc.

To clarify the reason why we restrict only to the three kind of local singularities of Figure 1 we shift to the one dimensional case where the analogous of simple polyhedra is the set of graphs whose local possible patterns are the one homeomorphic to the open interval and the one homeomorphic to an open  $Y$ -shaped graph; in this case the vertices are the vertices of the graph. A generic graph could contain some  $n$  (with  $n > 3$ ) valent vertex, so that the restriction to 3-valent graphs appears somewhat strong. Roughly speaking, the idea underlying this choice is that a  $n$ -valent vertex of a graph is “stable” only if  $n = 3$ : if we slightly perturb the position of an edge of the graph near, say, a 4-valent vertex, we can transform by a local modification the vertex into a pair of 3-valent vertices, passing from a  $X$ -shaped graph to a  $H$ -shaped one; if we try to repeat this operation for one of the two 3-valent vertices obtained that way, we cannot further modify the homeomorphism type of the graph. More formally (but we will not plug into these matters), each graph is simply-homotopic to a trivalent one. Analogously, in dimension 2, each polyhedron can be suitably perturbed to a new one whose singularities are those depicted in Figure 1.

The first example of simple polyhedron is a closed surface, whose points are all of the first type in Figure 1. An example containing only points of the first two types is the union of a sphere in  $\mathbb{R}^3$  and the disc spanned by a maximal circle in it and contained in the ball bounded by the sphere. This polyhedron has singular set composed by a circle and contains no vertices; its regions are three (open) discs.

A third example can be constructed by using the last one and gluing to it along another equator (different from the one already chosen) another disc, this time contained in the complement of the ball bounded by the initial sphere. The polyhedron we get has a singular set containing two 4-valent vertices (the intersections of the two maximal circles in the sphere used to construct it) and such that no edge of the graph has endpoints in the same vertex; the regions of the polyhedron are six discs.

In the above example we used a sphere embedded in  $\mathbb{R}^3$  but this was only for the sake of clarity: one the main properties of simple polyhedra is that they can be abstractly described by means of their combinatoric structure. To give an idea of how this can be performed we do this for the preceding example.

First, one identifies the structure of the singular set: in our case a 4-valent graph with two vertices connected by four edges. Then, one studies how each region is glued to the singular graph: in our case the boundary of each region passes exactly once over two different edges of the Singular set and zero times over the others. Moreover, for each pair of edges there is exactly one region whose boundary is the union of the two edges: there are 4 edges and hence 6 pairs of edges and exactly 6 regions. Hence, to reconstruct the polyhedron, it is sufficient to start from the 4-valent graph and glue the six regions over all the possible pairs of edges.

In the preceding case, we were treating a particular type of polyhedron: all its regions were discs and its singular graph was connected; such polyhedra are called *standard*. The above combinatorial reconstruction is applicable also in the general case of simple-polyhedra, but it is easier to describe in the standard case since there is no need of specifying the topology of each region to be glued to the singular graph. From now on, for the sake of brevity, we will use the word polyhedron for standard polyhedron, restricting the set of objects we will study; anyway, all our discussions will hold also for simple polyhedra in general.

Before ending this section, we define the notion of “mod 2 gleam” of a polyhedron  $P$ . Let  $D$  be a region of  $P$ , and let us consider the regular neighborhood of  $\partial D$  in  $P$ . For the sake of simplicity, we suppose that  $D$  is embedded in  $P$ , but the same arguments apply to the general case when  $\partial D$  runs more than once over an edge of  $Sing(P)$ . The regular neighborhood of a point  $p_0$  of  $\partial D$  different from a vertex of  $P$  is composed by a half plane (corresponding to  $D$ ) glued along a segment to a square in  $\mathbb{R}^2$  which is split in two components, corresponding to the other regions touching  $D$  along its boundary near  $p_0$ . Let us choose one of these components, and continue this choice by following  $\partial D$  until we reach a vertex of  $P$ . To extend this choice near the vertex, we delete from the neighborhood of the vertex the only region which locally touches  $D$  only in the vertex itself (there are six regions around a vertex and they are “opposite” in pairs): we obtain again a square divided in two by  $\partial D$  and we can continue our choice canonically (we recommend the reader to try and visualize it). After completing a whole loop around  $\partial D$ , we come back to the initial point  $p_0$  with a choice of one of the two components into which  $\partial D$  splits the neighborhood of  $p_0$  in  $P - D$ . If this second component is the one we choose initially then

we say that the mod 2 gleam of  $D$  is 0, otherwise it is 1. Acting similarly for each region of  $P$  we get a  $\mathbb{Z}_2$ -coloring on the regions of  $P$  called *mod 2 gleam* of  $P$ .

In the case of an embedded region  $D$ , the above description can be summarized as follows: consider the regular neighborhood of  $\partial D$  in  $P - \text{int}(D)$ ; if it collapses over a Moebius strip, then the mod 2 gleam of  $D$  is 1, otherwise it is 0.

### 3. POLYHEDRA IN 4-MANIFOLDS

In this section we investigate how a polyhedron  $P$  embedded in a 4-manifold  $M$  can be given an extra structure related to the topology of its regular neighborhood. Let us be more specific regarding the word “embedded”:

**Definition 3.1.** A polyhedron embedded in a smooth 4-manifold is said to be *locally flat* if for each point  $p$  of  $P$  there is a local chart  $(U, \phi)$  of the smooth atlas of  $M$  such that the image of  $P \cap U$  through  $\phi$  is contained in a 3-dimensional plane in which, up to a self-homeomorphism of the plane, it appears as one of the three local pictures of Figure 1 in  $\mathbb{R}^3 \subset \mathbb{R}^4$ , that is, around each of its points,  $P$  is contained in a 3-dimensional slice of  $M$  and in this slice it appears (up to homeo) as shown in Figure 1.

For the sake of brevity, from now on we will use the word “embedded” for “locally flat embedded”. The first question we ask ourselves is the following: can we reconstruct the regular neighborhood in a manifold  $M$  of a polyhedron  $P$  from its combinatorics? Before trying and ask this question in the 4-dimensional case, let us understand the easier 3-dimensional case, where  $P$  is a polyhedron embedded in an oriented 3-manifold  $N$ . In this case one sees that the  $\mathbb{Z}_2$ -gleam of each region of  $P$  is 0: indeed, roughly speaking, the regular neighborhood of a region of  $P$  (which is a disc) in  $N$  is  $D^2 \times [-1, 1]$  and so it is disconnected by the region; this would be false if the mod 2 gleam of the region was 1. Hence, the fact that a polyhedron embeds in a 3-manifold imposes some restrictions on its combinatorics. The truth is much deeper: the combinatorics of the polyhedron allows one to reconstruct the topology of its regular neighborhood in  $N$ . Indeed, by decomposing  $P$  in the local patterns shown in Figure 1, we can decompose its regular neighborhood in blocks as those of Figure 2 which can be reglued to each other according to the combinatorics of  $P$ . That way, a polyhedron with zero mod 2 gleam determines a 3-manifold collapsing onto it. It can be shown that any 3-manifold with non-empty boundary can be constructed that way, and so the study of three manifolds can be attacked by means of particular polyhedra; this is the basic idea underlying the theory of spines of 3-manifolds.

Let us now skip to the 4-dimensional case and, for the sake of simplicity, choose  $P$  to be homeomorphic to  $S^2$ . Suppose that  $P$  is embedded in a smooth, oriented 4-manifold  $M$ : it is possible to reconstruct its regular neighborhood using only its topology? The answer is “no” since the regular neighborhood of a sphere (and more generally of a surface) is determined by the topology of the surface and by its self-intersection number in the manifold. To state it differently, the regular neighborhood of a surface in a 4-manifold is diffeomorphic to the total space of a disc bundle over the surface (its normal bundle) and the Euler number of this bundle is a necessary datum to reconstruct its topology.

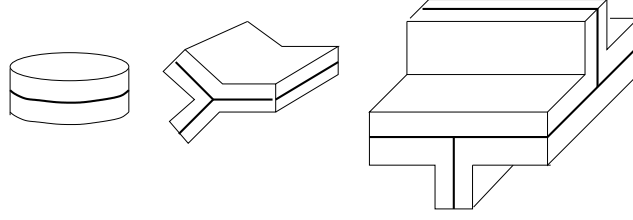


FIGURE 2. The three type of blocks used to thicken a spine of a 3-manifold.

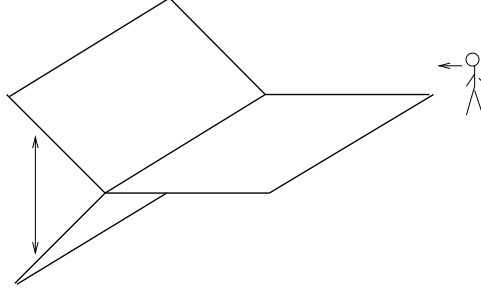


FIGURE 3. The picture sketches the position of the polyhedron in a 3-dimensional slice of the ambient 4-manifold. The direction indicated by the vertical double arrow is the one along which the two regions touching the horizontal one get separated.

Hence, we see that to codify the topology of the regular neighborhood of  $P$  in  $M$  we need to decorate  $P$  with some additional information; when  $P$  is a surface, the Euler number of its normal bundle is a sufficient datum.

More generally the following holds:

**Proposition 3.2.** *Let  $P$  be a polyhedron embedded in an oriented smooth 4-manifold  $M$ . There is a coloring of the regions of  $P$  with values in the half integers  $\frac{1}{2}\mathbb{Z}$  canonically induced by its embedding in  $M$ . This coloring is called the gleam of  $P$  in  $M$ .*

*Proof of 3.2.* The construction we will describe is similar to the one used to define the mod 2 gleam of a polyhedron. For the sake of simplicity let again  $D$  be a region of  $P$  whose boundary is embedded in  $Sing(P)$ . Let  $p_0$  be a point of  $\partial D$  and let us consider the three dimensional slice  $B^3$  of  $M$  into which a regular neighborhood of  $p_0$  in  $P$  is sitting (it exists since by hypothesis  $P$  is locally flat in  $M$ ). Let us fix a auxiliary riemannian metric on  $M$  and consider the orthogonal direction to  $D$  in  $B^3$ ; by construction, it coincides with the direction along which the two other regions touching  $\partial D$  on  $p_0$  get separated (see Figure 3).

We can extend the definition of this direction to the whole  $\partial D$ , in the same way we did to get the mod 2 gleam, obtaining a continuous choice of orthogonal directions to  $D$  in

$M$  along  $\partial D$ . This choice represents a section of the bundle of orthogonal directions to  $D$  in  $M$ , whose fiber is  $S^1$ . The obstruction to extend this section to the whole  $D$  can be canonically identified with an integer: indeed it represents an element of  $H^2(D, \partial D; \pi_1(S^1))$  and if we fix an orientation on  $D$  then also the fiber gets oriented since  $M$  is oriented, hence giving an identification of  $H^2(D, \partial D; \pi_1(S^1))$  with  $H^2(D, \partial D; \mathbb{Z}) = \mathbb{Z}$ ; this identification does not change if we revert the orientation of  $D$ , since also the orientation of the fiber changes. Hence, if we divide by 2 the number obtained through this identification, we get an element of  $\frac{1}{2}\mathbb{Z}$  which we will call the gleam of  $D$ , and which represents the Euler number of the normal bundle of  $D$  in  $M$ . It is worth note that the Euler number of the normal bundle of an embedded surface equals the self-intersection number of the surface, so that the gleam represents a version of the self intersection number “localized” on the regions of the polyhedron.

The above construction can be performed analogously in the case when  $D$  is not embedded in  $P$ ; that way, we can define a gleam over each region of  $P$ . Note that, by construction, the gleam of a region of  $P$  is non-integer (an odd multiple of  $\frac{1}{2}$ ) if and only if its mod 2 gleam is 1.

3.2

#### 4. FROM POLYHEDRA WITH GLEAMS TO 4-MANIFOLDS

In the preceding section we showed that any polyhedron embedded in a 4-manifold can be equipped with a gleam. This produces a map from the pairs  $(M, P)$  of smooth, oriented and compact manifolds containing embedded polyhedra into the set of polyhedra equipped with gleams  $(P, gl)$  where a gleam is a  $\frac{1}{2}\mathbb{Z}$  coloring of the regions of  $P$  such that the color of a region is non integer if and only if its mod 2 gleam is 1. We also stated that if  $P$  is a surface then its gleam is the Euler number  $e$  of the normal bundle of the surface in  $M$  and hence the pair  $(P, e)$  is sufficient to reconstruct the regular neighborhood of  $P$  in  $M$ . This is true in general as proved by Turaev in [17]:

**Theorem 4.1** (Reconstruction map). *Let  $(P, gl)$  be a polyhedron equipped with gleams; there exists a canonical reconstruction map associating to  $(P, gl)$  a pair  $(M_P, P)$  where  $M_P$  is a smooth, compact and oriented 4-manifold containing an embedded copy of  $P$  over which it collapses and such that the gleam of  $P$  in  $M_P$  coincides with the initial gleam  $gl$ . The pair  $(M_P, P)$  can be explicitly reconstructed from the combinatorics of  $P$  and from  $gl$ . Moreover, if  $P$  is a polyhedron embedded in a smooth and oriented manifold  $M$  and  $gl$  is the gleam induced on  $P$  as explained in the preceding section, then  $M_P$  is diffeomorphic to a regular neighborhood of  $P$  in  $M$ .*

The above theorem is the key tool of the theory of shadows of 4-manifolds: indeed by means of this result, one is allowed to study 4-manifolds in a purely combinatorial way, using only polyhedra equipped with gleams.

*Proof of 4.1.* The idea of the proof is to decompose  $P$  in blocks as those of Figure 1, thicken them to canonical blocks, and then describe how to glue these blocks according to

the combinatorics of  $P$  and to its gleam. The shape of the basic blocks is easily understood: consider the blocks used to reconstruct 3-manifolds from their spines and take their products with  $[-1, 1]$ . Then the block corresponding to a region is  $D^2 \times D^2$  and the blocks corresponding to edges and vertices of  $Sing(P)$  are the products respectively of the second and third block of Figure 2 with a segment  $[-1, 1]$  representing “the fourth dimension”.

We first reconstruct the regular neighborhood in  $M_P$  of the subpolyhedron  $S$  of  $P$  coinciding with the regular neighborhood of  $Sing(P)$  in  $P$ ; to do this, we use the blocks corresponding to vertices and edges of  $P$ . The combinatorics of  $P$  forces us to glue these blocks to each other and leaves us only one choice: whether gluing them by reverting or not the  $[-1, 1]$  factor of the fourth dimension of the blocks. If  $P$  contains  $n$  vertices, we have to connect the  $n$  blocks corresponding to these vertices through the  $2n$  blocks corresponding to the edges of the 4-valent graph  $Sing(P)$ . Consider a maximal subtree  $T$  of  $Sing(P)$ , and glue the blocks corresponding to the vertices and the edges of  $T$  according to the combinatorics of  $S$  choosing arbitrarily the gluing on the level of the  $[-1, 1]$  factors, that is choosing arbitrarily whether identifying the  $[-1, 1]$  fibers by the identity or the multiplication by  $-1$ . The manifold constructed that way is  $B^4$  and contains a properly embedded copy of a regular neighborhood of  $T$  in  $S$ . Only  $n + 1$  blocks corresponding to the edges of  $Sing(P) - T$  are now left to be glued according to the combinatorics of  $S$ . For each of these blocks, there is only one way of gluing the  $[-1, 1]$  fiber to get an orientable 4-manifold; moreover, at the end of this construction, the manifold  $H$  obtained is diffeomorphic to the regular neighborhood of a 4-valent graph ( $Sing(P)$ ) in  $\mathbb{R}^4$ .

Note that since  $H$  admits an orientation reversing diffeomorphism corresponding to the product by  $-1$  along the fiber  $[-1, 1]$  of the blocks we used to construct it, then it is canonically oriented. Moreover, by construction,  $H$  fibers over a (not necessarily orientable) 3-manifold  $L$  whose spine is  $S$ ; the fiber is exactly the  $[-1, 1]$  factor of the blocks used to construct  $H$ . To construct  $L$  it is sufficient to repeat the above construction of  $H$  using the 3-dimensional blocks of Figure 2 without taking the product with  $[-1, 1]$ ;  $L$  is not necessarily orientable and, since its construction does not deal with the regions of  $P$ , no assumption on the gleams of the regions of  $P$  is necessary to construct  $L$  or  $H$ . Finally, a copy of  $S$  is properly embedded in  $L$ ; then it is embedded in a locally flat way in  $H$  so that its boundary is contained in  $\partial H$ . Each boundary component of  $S$  corresponds by construction to the boundary component of a region of  $P$ , hence to finish our construction we have to glue along  $\partial S \subset \partial H$  the blocks corresponding to the regions of  $P$ .

Let us show how to do this for a single region  $Y$  of  $P$ . The block corresponding to  $Y$  is  $Y \times D^2$  and, since by hypothesis  $Y$  is a disc, it is a 4-ball with a prescribed 2-handle structure,  $Y$  being the core of the handle. To describe how to glue this 2-handle to  $H$  we:

- (1) identify the image of  $\partial Y$  in  $\partial H$  through the gluing map;
- (2) identify the image in  $\partial H$  through the gluing map of the canonical framing of  $\partial Y$ , that is of the curve  $\partial Y \times \{1\}$ .

The first step is straightforward:  $\partial Y$  has to be identified with the boundary component  $s$  of  $S$  contained in  $\partial H$  corresponding to  $Y$ . To perform the second step, we have to identify a canonical framing on  $s$ ; this can be done since  $H$  fibers over  $L$  with fiber  $[-1, 1]$ .



Indeed, while assembling  $H$ , we glued blocks having the structure of products between 3-dimensional blocks and  $[-1, 1]$  and we always respected this product structure up to the orientation of the  $[-1, 1]$ -fiber. Hence each boundary component of  $S \subset H$  (and in particular  $s$ ) is automatically equipped with the framing given by the  $[-1, 1]$ -direction; the only delicate point is that the fiber over  $s$  of the projection of  $H$  onto the  $L$  could in some cases be a Moebius strip instead of an annulus.. If it is an annulus, we have a genuine framing on  $s$  and hence, to attach the 2-handle corresponding to  $Y$ , is sufficient to say how many times the image of the canonical framing of  $\partial Y$  in  $Y \times D^2$  twists around  $s$  in  $\partial H$  with respect to this framing; since  $H$  is oriented, an integer number is then sufficient to encode this information, and we use the gleam of  $Y$ . Indeed, the gleam of  $Y$  is integer exactly when the inverse image in  $H$  of the curve  $s$  through the projection along the  $[-1, 1]$ -fiber is an annulus.

We are left with the case when the inverse image of  $s$  in  $H$  through the projection along the  $[-1, 1]$ -direction is a Moebius strip  $M_s$ . In this case, the gleam of  $Y$  is half integer; we then add a positive half twist to  $M_s$  around  $s$  (recall that  $H$  is oriented), obtaining an annulus whose core is  $s$  and hence a framing on  $s$ . We now perform the construction of the preceding case using as new gleam for  $Y$  the integer given by the initial gleam of  $Y$  minus  $\frac{1}{2}$ .

Performing the above construction for all the regions of  $P$  we get a 4-dimensional “neighborhood” of  $P$ ; it can be checked that the gleam induced on  $P$  by this neighborhood coincides with the initial gleam  $gl$ . 4.1

**Example 4.2.** Let  $P$  be a spine of an orientable 3-manifold  $N$ ; in particular its mod 2 gleam has to be zero. By performing the construction above, using as gleam on  $P$  the 0 gleam over all the regions, we get the manifold  $N \times [-1, 1]$ .

**Remark 4.3.** All the 4-manifolds obtained by “thickening” the polyhedra equipped with gleams as in the proof of Theorem 4.1 admit a handle decomposition containing no handles of index greater than 2. It can be shown that also the reverse holds: any manifold admitting such a handle decomposition can be obtained by applying Theorem 4.1 to a suitable polyhedron equipped with gleams (see [5]).

**Definition 4.4.** A polyhedron equipped with gleam  $(P, gl)$  is a *shadow* of a 4-manifold  $M$  if  $M$  is diffeomorphic to the thickening  $M_P$  of  $P$  obtained through Theorem 4.1.

The above remark shows that shadows can be used to describe combinatorially only a subset of all the smooth 4-manifolds not including closed ones. To obviate to this apparent weakness of the theory, let us recall the following result due to F. Laudenbach and V. Poenaru proved in [12]:

**Theorem 4.5.** *Let  $M$  be an oriented, smooth and compact 4-manifold with boundary equal to  $S^3$  or to a connected sum of copies of  $S^2 \times S^1$ . Then there is only one (up to diffeomorphism) closed, smooth and oriented 4-manifold  $M'$  obtained by attaching to  $M$  only 3 and 4-handles.*

Roughly speaking, the above result states that when a manifold is “closable” then it is in a unique way (up to diffeomorphism). This allows us to describe all the closed 4-manifolds by means of polyhedra with gleams: indeed given a closed manifold equipped with an arbitrary handle decomposition, considering the union of all handles of index strictly less than 3 we get a new manifold  $M$  which admits a shadow and can be described combinatorially as explained above. The initial manifold can be then uniquely recovered from  $M$  because of Theorem 4.5. We can then give the following definition:

**Definition 4.6.** A polyhedron with gleams  $(P, gl)$  is a *tunnel shadow* of a closed 4-manifold  $X$  if and only if  $X$  can be obtained by attaching 3 and 4-handles to the 4-manifold  $M_P$  obtained from  $P$  through the reconstruction map of Theorem 4.1.

The name “tunnel shadow” comes from the fact that the inverse of attaching a 3-handle is the operation of digging a tunnel along a properly embedded arc. We will see later how to construct a shadow of the complement of a properly embedded arc in the thickening  $M_P$  of a shadow  $P$ .

**Example 4.7.** Let  $P$  be equal to a 2-sphere and let its gleam be  $-1$ . The thickening of  $(P, gl)$  is the total space of the disc bundle over  $S^2$  with Euler number  $-1$ , whose boundary is  $S^3$ . This space is the complement of a point in  $\mathbb{CP}^2$ , and, because of Theorem 4.5,  $(P, gl)$  is a tunnel shadow of  $\mathbb{CP}^2$ .

## 5. SHADOWS AND BASIC MOVES

In this section we outline the list of basic modifications of polyhedra with gleams called *moves* which allow one to produce new shadows of the same manifold from a given one.

From now on, let  $(P, gl)$  be a shadow of a 4-manifold  $M$  into which it is embedded through the construction of the proof of Theorem 4.1. The first three moves we describe are called *shadow equivalences*; they modify the position of  $P$  in  $M$  producing a new shadow  $P'$  embedded in  $M$  which differs from  $P$  only inside a 4-ball.

These moves are shown in Figure 4; let us by the moment forget about the letters written in the figure. To visualize these moves, imagine that a region of  $P$  is slid over some other regions, producing a polyhedron which is different from the initial one only in a contractile subpolyhedron (the one pictured in the left part of the figure). These moves are called respectively  $1 \rightarrow 2$ ,  $0 \rightarrow 2$  (or lune) and  $2 \rightarrow 3$ -moves, because of their effect on the number of vertices of the polyhedra. Since the  $1 \rightarrow 2$ -move is the trickiest to understand, we concentrate first on the  $0 \rightarrow 2$ -move (also called finger move or lune move); this move acts in a 4-ball contained in  $M$  and containing the part of  $P$  shown in the left part of the picture. After the move, we substitute this part of  $P$  with the new one drawn in the right part of the figure and we keep the 4-ball unchanged, obtaining a shadow  $P'$  of the same 4-manifold  $M$ . One can imagine the move as a sliding over the horizontal regions of the region which in the left part of the figure is the upper-left; this move can be performed in a 3-dimensional slice of the 4-ball in  $M$ ; hence we say that the  $0 \rightarrow 2$ -move has a 3-dimensional nature. The same holds for the  $2 \rightarrow 3$ -move, also called Matveev-Piergallini move: although apparently more complicated than the  $0 \rightarrow 2$ -move, it can be seen that it

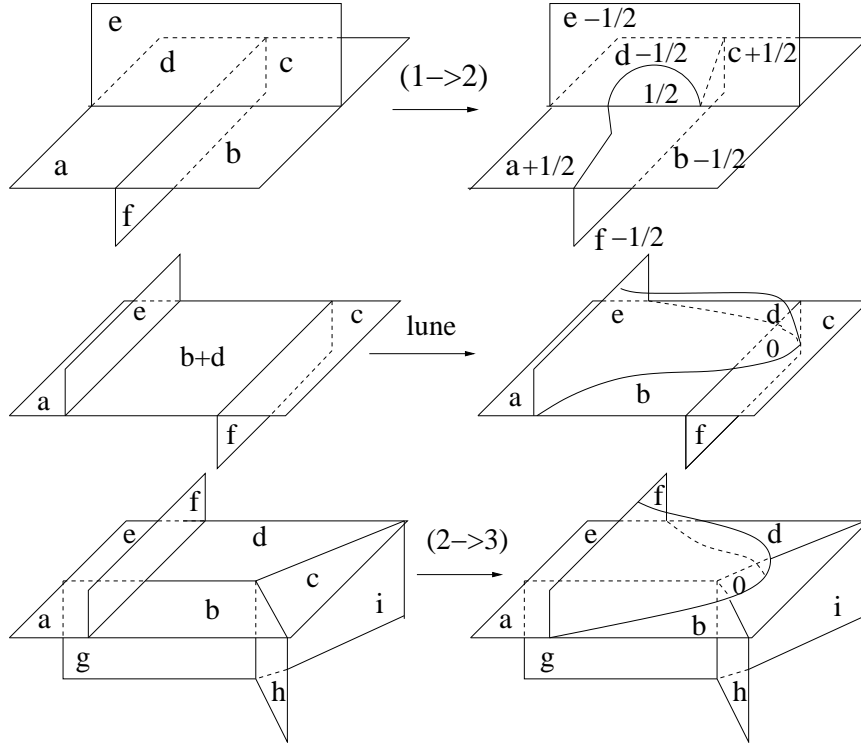


FIGURE 4. The three shadow equivalences.

is a version of the  $0 \rightarrow 2$ -move in which the sliding region passes over a vertex. The same comments as above apply to this case.

The case of the  $1 \rightarrow 2$ -move is different; this move applies to a neighborhood of a vertex and lets a region (the vertical lower one in the left part of the figure) slide over the vertex producing a new one. It is a good exercise to try to visualize this sliding: it cannot be performed in  $\mathbb{R}^3$ ; if we want the sliding region not to touch in its interior the other regions near the vertex, then it is necessary to use the fourth dimension. Hence we say that this move has 4-dimensional nature; it has the effect of modifying  $P$  but not  $M$ . Note that, even if the move “needs the fourth dimension”, its result is still a locally flat polyhedron in  $M$ .

Of course, the above comments apply also to the inverses of the moves, that is to the moves which in the figure go from right to left. Let us now clarify the meaning of the letters written on the regions of the polyhedra in the figure: they represent the gleams of the regions; note that only the  $1 \rightarrow 2$ -move changes the gleams of the regions near the vertex over which it is applied. Each move represents a modification of the embedded polyhedron, and hence produces a new polyhedron whose gleams (induced by

the embedding in the ambient manifold as explained in Proposition 3.2) can differ from those of the initial polyhedron; moreover these gleams can change only on the regions which appear in the pictures, since otherwise the position of the regions in  $M$  is left unchanged by the move.

All these comments can be resumed in the following proposition:

**Proposition 5.1.** *Let  $P$  be a shadow of a 4-manifold  $M$  and let  $P'$  be obtained from  $P$  through the application of any sequence of  $1 \rightarrow 2$ ,  $0 \rightarrow 2$  and  $2 \rightarrow 3$  embedded moves (and possibly their inverses), then  $P'$  is a shadow of  $M$ . Moreover, the gleam of  $P'$  can be reconstructed from the gleam of  $P$  by following step by step the changes on the gleams induced by the moves of the sequence.*

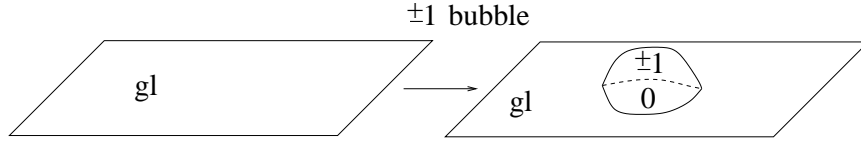


FIGURE 5. The  $\pm 1$ -bubble moves: the gleam of the region over which the move is applied is left unchanged. A new 0-gleam disc is created and a  $\pm 1$ -gleam disc is attached to the boundary of the latter disc. The 0-bubble move, creates a 0-gleam disc in place of the  $\pm 1$ -gleam one.

The above moves change the polyhedron but do not change the ambient manifold; we now describe two moves which change also the manifold and which are useful in a number of applications we will outline in the next sections: the 0-bubble move and the  $\pm 1$ -bubble move. On the level of naked polyhedra they are identical and shown in Figure 5; they differ on the level of the gleams of the polyhedron after the move. The 0-bubble move applied to a polyhedron  $P$  produces a polyhedron  $P'$  by attaching a 0-gleam disc to a 0-gleam disc contained in a region of  $P$  and leaving unchanged the gleams of the other regions of the polyhedron;  $P'$  is not standard, but the construction of Theorem 4.1 can be generalized to simple polyhedra equipped with gleams. Let  $M_P$  and  $M_{P'}$  be the 4-manifolds determined respectively by  $P$  and  $P'$  as explained in Theorem 4.1; it can be checked that  $M_{P'}$  is diffeomorphic to the complement in  $M_P$  of a properly embedded arc, that is, it is obtained by tunneling  $M_P$ . Conversely,  $M_P$  can be recovered from  $M_{P'}$  by attaching a 3-handle whose cocore corresponds to that arc.

Let us now describe the effects of the  $\pm 1$ -bubble moves. These moves are identical except the sign chosen; we concentrate on the  $-1$ -bubble move. Its application to a polyhedron  $P$  produces a polyhedron with gleams  $P'$  by attaching a  $-1$ -gleam disc over a 0-gleam disc in a region of  $P$  and leaving the other gleams unchanged. It can be checked that if  $M_P$  is the manifold determined by  $P$  then the one determined by  $P'$  is  $M_{P'} = M_P \# \mathbb{CP}^2$ , and, in particular,  $\partial M_P = \partial M_{P'}$ ; analogously the application of a 1-bubble move produces a shadow of  $M_P \# \overline{\mathbb{CP}}^2$ .

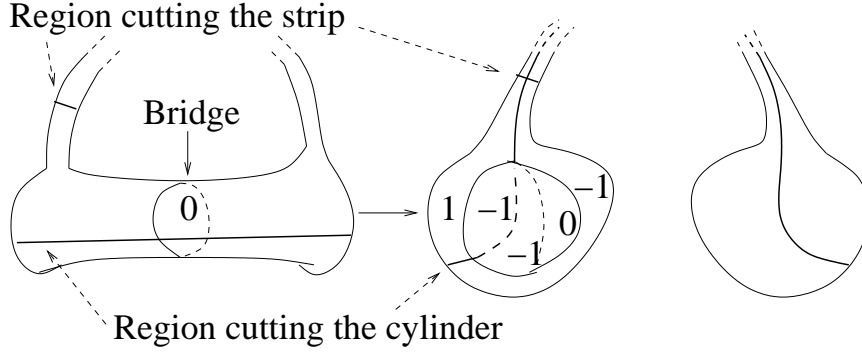


FIGURE 6. In the left part of the figure we show the pattern to which the trading-move applies. It is constituted of a cylinder cut by a zero gleam disc and of a (possibly twisted) strip connecting the two ends of the cylinder. Some strands (corresponding to attaching curves of regions of the shadow) can pass over the cylinder or cut the strip. In the right part we show the effect of the trading-move: the cylinder disappears, its two ends are capped off with two zero gleam discs and, in one of them, a little 0-gleam two disc is attached. The strands which before the move run across the cylinder, are now completed along the strip and linked with the little zero gleam disc as shown in the drawing.

The last move we introduce is called the “trading”-move and is visible in Figure 6. Unlike the above moves, this move does not start from a contractible sub-polyhedron of the initial polyhedron; hence we say that it is “non-local”: one can apply it only when a pattern in the polyhedron which coincides with the left part of the figure is found. The move dramatically changes the topology of the polyhedron: in particular the fundamental group changes. For the sake of simplicity, in the figure we draw a version of the trading move in the case of a non standard, simple polyhedron; up to the application of a suitable number of  $0 \rightarrow 2$ -moves, the polyhedron can be transformed into a standard one and the move can be described also in the standard setting but the drawings are more complicated and we omit them here.

**Proposition 5.2** ([17]). *Let  $P$  be an integer shadowed polyhedron and let  $P'$  be the polyhedron obtained by applying a trading-move (or its inverse) to  $P$ . Then the manifold  $M_{P'}$  is obtained from  $M_P$  by doing surgery along a closed curve  $C$  embedded in the interior of  $M_P$  (respectively, on an embedded sphere whose self intersection number is zero), and then, in particular,  $\partial M_P$  and  $\partial M_{P'}$  are diffeomorphic.*

*Proof of 5.2.* We limit ourselves to rough motivations for the above proposition; see [17] for details. We first analyze the portion of manifold described in Figure 6 before the move. We call *bridge* the part of the polyhedron  $P$  formed by the curved cylinder together with the central disc with zero gleam in the left part of Figure 6. It can be checked that there

exists a handle decomposition of  $M_P$  such that the bridge is contained in a 1-handle and the two ends of the bridge are contained in the two ends of the 1-handle. The other regions which in the figure run along the cylinder correspond to 2-handles whose attaching curves pass through the 1-handle. So, from the point of view of the Kirby calculus, the left part of Figure 6 corresponds to the left part of Figure 7, where the strand running from one attaching sphere of the 1-handle to the other one corresponds to the attaching curve which in Figure 6 runs across the bridge.

Now, let  $C$  be a closed curve inside  $M_P$  which passes once over the bridge in Figure 6 and then gets closed the strip connecting the two ends of the cylinder; surgering along  $C$  produces a 4-manifold which, in place of the one-handle determined by the bridge, contains a two-handle whose attaching curve is an unknot in  $\partial M_P$  and such that the attaching curve of every 2-handle which passed over the bridge, passes inside the disc bounded by the unknot (recall that surgering along a curve in a 4-manifold consists of substituting the regular neighborhood of the curve with a copy of  $S^2 \times D^2$ ).

It can be checked, but we will not do it here, that this move corresponds to the move which in Kirby calculus, given a presentation of a 3-manifold, allows one to exchange a one handle with a zero framed two handle represented by an unknot and linked with all the two handles passing over the one handle, obtaining a presentation of the same 3-manifold (see the right part of Figure 7).

5.2

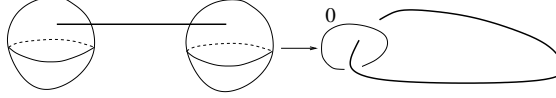


FIGURE 7. In the left part of the figure, we show an example of 1-handle in a Kirby calculus presentation of a 4-manifold over which an attaching curve of a 2-handle passes. In the right part we show the effect of trading the 1-handle for a 2-handle.

We resume the facts exposed in this section in the following theorem. For a complete account we refer to [17].

**Theorem 5.3.** *Let  $P$  and  $P'$  be two integer shadowed polyhedra such that  $P'$  is obtained from  $P$  by applying a move of the following types:*

- (1)  $2 \rightarrow 3$ -move or its inverse;
- (2) *finger-move* or its inverse;
- (3)  $1 \rightarrow 2$ -move or its inverse.

*Then  $M_P$  and  $M_{P'}$  are diffeomorphic by an orientation preserving diffeomorphism.*

*If  $P'$  is obtained from  $P$  by a  $\pm 1$ -bubble move then  $M_{P'}$  is diffeomorphic to the boundary connected sum of  $M_P$  and a punctured  $\mathbb{CP}^2$  or  $\overline{\mathbb{CP}^2}$  respectively, hence  $\partial M_{P'}$  is diffeomorphic (by an orientation preserving diffeomorphism) to  $\partial M_P$ . If  $P'$  is obtained from  $P$  by*

a 0-bubble move then  $M_{P'}$  is diffeomorphic to the boundary connected sum of  $M_P$  and  $S^2 \times D^2$ , and then, in particular  $\partial M_{P'} = \partial M_P \# S^2 \times S^1$ . Finally, if  $P'$  is obtained from  $P$  by the application of a trading move, then  $M_{P'}$  is obtained from  $M_P$  by doing a surgery along the curve which passes once over the bridge and once over the cylinder of Figure 6; in particular  $\partial M_{P'} = \partial M_P$ .

## 6. SOME APPLICATIONS AND QUESTIONS

In this section we briefly recall some results and open questions. The first one surges once one proves Theorem 4.1 and is the following:

**Question 6.1.** *Given two shadows  $P$  and  $P'$  of the same manifold, can they be connected by a sequence of suitable moves, for instance shadow equivalences (see the beginning of Section 5)?*

A first answer to the above question is due to Turaev [17]:

**Theorem 6.2.** *Let  $P$  and  $P'$  be two shadows of the same 4-manifold  $M$ , then they can be connected by a sequence of shadow equivalences, 0-bubble moves and their inverses. Moreover the sequence always contains a number of 0-bubbles equal to the number of inverse 0-bubble moves.*

In the above result there is a disturbing aspect: the sequence of moves connecting  $P$  and  $P'$  can contain 0-bubble moves and can produce at a certain step a polyhedron which is not a shadow of  $M$ , hence in general the sequence does not pass only through shadows of  $M$ . Moreover the converse of the above theorem is not yet proved:

**Question 6.3.** *Let  $P$  and  $P'$  be obtained from each other by a sequence of shadow equivalences and 0-bubble moves containing a number of 0-bubble moves equal to the number of their inverses. Is it true that the manifold  $M$  and  $M'$  determined by  $P$  and  $P'$  respectively are diffeomorphic?*

The author in [5] proved that the answer to the above question is “yes” when  $P$  is (and hence  $P'$ ) is simply connected; the positive answer in general is still a conjecture.

The following question can be thought of as an embedded version of the Andrews-Curtis conjecture:

**Question 6.4.** *Let  $P$  and  $P'$  be shadows of the same manifold; can one connect them by a sequence of shadow equivalences? (No bubble moves!)*

By Matveev and Piergallini’s results ([13], [14]), the analogous question in dimension 3 has a positive answer. The 4-dimensional case remains open; it has to be pointed out here that, although apparently the above question is strictly related to the Andrews-Curtis conjecture, by the moment no formal relation between the two problems has been proved.

Theorem 6.2 can be re-read in terms of tunnel shadows of closed 4-manifolds, and it then states that any two shadows of a closed 4-manifold are connected by a sequence of shadow equivalences and 0-bubble moves. In this case, it represents a genuine calculus in the standard sense.

## 7. SHADOWS OF 3-MANIFOLDS

In this section we will briefly recall the notion of a shadow of a 3-manifold and give an overview of some recent results on the subject.

**Definition 7.1.** Let  $(P, gl)$  be a shadow of a 4-manifold  $M$ ; then  $(P, gl)$  is a shadow of the three-manifold  $N = \partial M$ .

It can be proved that any closed and oriented 3-manifold admits a shadow, and even a simply connected one; moreover, the application of any shadow equivalence or  $\pm 1$ -bubble move to a shadow of a 3-manifold produces a shadow of the same manifold. In general, the following was proved in [8]:

**Theorem 7.2** (Calculus for shadows of 3-manifolds). *Any two simply connected shadows of the same closed and oriented 3-manifold can be connected by a suitable sequence of shadow equivalences,  $\pm 1$ -bubble moves and their inverses.*

The above result is an analogue of the Kirby calculus in the world of shadows of 3-manifolds, where the blow-up move is substituted by the  $\pm 1$ -bubble moves; its pretty feature is that it is based only on local moves.

A new notion of complexity of 3-manifolds based on shadows is defined in [9]:

**Definition 7.3.** Let  $N$  be a closed and oriented 3-manifold. The *shadow complexity* of  $N$  is the minimal number of vertices of a (possibly non standard) shadow of  $N$ .

**Example 7.4.** As shown in Example 4.7, the polyhedron  $S^2$  equipped with gleam  $-1$  is a shadow of  $S^3$  and hence the shadow complexity of  $S^3$  is zero.

Strong relations have been found between shadow complexity and hyperbolic geometry of 3-manifolds. The following Theorem, to appear in [9], summarizes some of them:

**Theorem 7.5.** *There exist universal positive constants  $h_1$  and  $h_2$  such that for any hyperbolic, complete 3-manifold  $N$  with finite volume the following inequalities hold:  $h_1 \text{Vol}(N) \leq sc(N) \leq h_2 (\text{Vol}(N))^2$ , where  $sc(N)$  is the Shadow Complexity of  $N$  and  $\text{Vol}(N)$  is its hyperbolic volume.*

Moreover, in the same paper we prove that each 3-dimensional graph manifold admits a shadow containing no vertices.

These results suggest that a study of hyperbolic manifolds by means of shadows can be attempted; to that purpose, one of the first questions to be answered is the following:

**Question 7.6.** *Given a polyhedron with gleams  $(P, gl)$ , what are the necessary and sufficient conditions on  $(P, gl)$  ensuring that the 3-manifold determined by the shadow is irreducible, or atoroidal or hyperbolic?*

A clear and definitive combinatorial condition answering the above question is still missing. Partial answers have been obtained in [4]: for instance, it has been proved that if a 3-manifold admits a standard shadow whose gleams are all greater than 7 in absolute value, then the manifold admits a metric with negative sectional curvature, and, consequently, is aspherical and atoroidal.



## 8. A GLANCE ON BRANCHED SHADOWS

Besides the open problems exposed in the preceding sections, which, in part, are internal to the theory of shadows, a study of additional structures and invariants of 4-manifolds can be attempted by means of shadows. In this section we summarize the results proved by the author in [5] (and which will be exposed in a self contained way in [6] and [7]) in this direction and based on the notion of branched shadow.

Given a simple polyhedron  $P$  we define the notion of *branching* on it as follows:

**Definition 8.1** (Branching). A branching  $b$  on  $P$  is a choice of an orientation for each region of  $P$  such that for each edge of  $Sing(P)$ , the orientations induced on the edge by the regions containing it do not coincide.

Not all the simple polyhedra admit a branching and, on the contrary, there are some which admit many different ones. So, we will say that a polyhedron is *branchable* if it admits a branching and we will call *branched polyhedron* a pair  $(P, b)$  where  $b$  is a branching on the polyhedron  $P$ .

**Definition 8.2.** Let  $(P, gl)$  be a shadow of an oriented 4-manifold  $M$ ;  $P$  is said to be *branchable* if the underlying polyhedron is. We call *branched shadow* of  $M$  the triple  $(P, gl, b)$  where  $(P, gl)$  is a shadow and  $b$  is a branching on the underlying polyhedron. When this does not cause any confusion, we do not specify the branching  $b$  nor the gleam  $gl$  and we will simply write  $P$ .

A branching on a simple polyhedron allows us to smoothen its singularities and equip it with a smooth structure as shown in Figure 8.

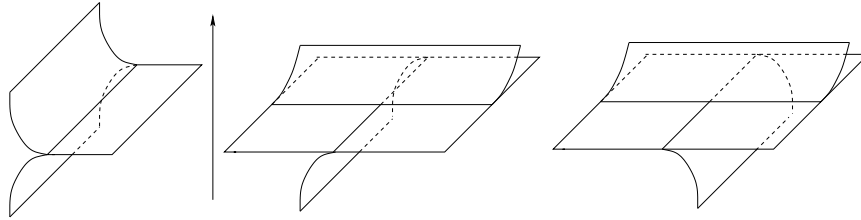


FIGURE 8. How a branching allows a smoothing of the polyhedron.

Using the orientation and the smooth structure of the shadow, in [6] (see also [5]), we will show how to equip the regular neighborhood of a shadow in a 4-manifold with a pair mutually transverse distributions of oriented 2-planes (roughly speaking, the planes which in Figure 8 are horizontal and their orthogonal complements). Then, in the same paper using this pair of distributions of oriented 2-planes, we show how to fix a homotopy class of almost complex structures on the regular neighborhood of the shadow, and hence a  $Spin^c$ -structure on this manifold. The result to appear in [6] can be summarized as follows:

**Theorem 8.3.** *Let  $M$  be a 4-manifold admitting a shadow and let  $s$  be a  $\text{Spin}^c$ -structure on  $M$ . There exists a branched shadow  $P$  of  $M$  representing the  $\text{Spin}^c$ -structure  $s$ . Stated differently, given a homotopy class of almost complex structures  $[J]$  on  $M$ , there exists a branched shadow of  $M$  representing the homotopy class  $[J]$  of almost complex structures on  $M$ .*

Following the line of the last part of Theorem 8.3, it can be studied how branched shadows can be used to find genuine complex structures on their thickening. In [6], the problem is attacked through techniques that are similar to those already used for real surfaces in complex manifolds by Bishop in [1] and Harlamov-Eliashberg in [11], based on counting the number of complex elliptic and hyperbolic points. The result we obtain can be summarized as follows:

**Theorem 8.4** ([6]). *Let  $P$  be a branched shadow reconstructing the manifold  $M$  equipped with the almost complex structure  $J$ . There exists a genuine complex structure  $J'$  homotopic to  $J$  such that  $P$  contains no  $J'$ -complex points of negative type, that is,  $P$  contains no points where its tangent plane is  $J'$ -complex but oriented differently by the branching on  $P$  and its complex structure induced by  $J'$ .*

As a natural development of the above result, using Gompf-Eliashberg's constructions, combinatorial sufficient conditions on a branched shadow ensuring that its thickening admits a Stein domain structure will be studied in [7] (see also [5], Chapter V). The conditions we will find are based on the definition (which we will not recall here) for each region of a branched polyhedron of an integer coefficient called the Euler number which is easily calculable from the combinatorial structure of the polyhedron. The main result, which will be exposed in a self-contained way in [7], is summarized in a simplified version by the following:

**Theorem 8.5** ([7]). *Let  $P$  be a branched shadow of a four manifold  $M$ . There exist constants  $k_i$  such that if, for each region  $R_i$  of  $P$  it holds  $\text{Eul}_i + \text{gl}_i \leq k_i$  where  $\text{gl}_i$  is the gleam of  $R_i$  and  $\text{Eul}_i$  is its Euler number, then  $M$  admits a Stein structure whose underlying complex structure belongs to the homotopy class of almost complex structures determined by  $P$ .*

The conditions found in the above theorem have been exploited in [5] to explicitly exhibit minimal genus representatives of some particular homology classes of surfaces in 4-manifolds:

**Corollary 8.6.** *Let  $P$  be a branched shadow satisfying the hypotheses of Theorem 8.4 and such that  $\text{gl}_i \geq 0$  for every  $i$ , and let  $[S] \in H_2(P; \mathbb{Z})$  be a homology class of the form  $\sum_i n_i R_i$  where  $R_i$  are the regions of  $P$  and  $n_i$  are non negative integers. Then there is an explicit construction of an oriented embedded representative  $S' \subset M$  of  $[S]$  having the lowest possible genus in its homology class.*

Moreover, using Theorem 8.4 on branched shadows equipped with zero gleams on every region, the following result, which was conjectured by Benedetti and Petronio in [2] is proved:

**Theorem 8.7.** *Let  $P$  be a branched spine of an oriented 3-manifold  $N$  such that the Euler number of each region of  $P$  is non positive, then the homotopy class of oriented 2-planes encoded by the branching of the spine on its 3-dimensional thickening contains a representative distribution which is a tight contact structure.*

The above results are likely to be non optimal and further study is required to fully understand the potential of the theory of branched shadows.

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